

Pure quantum states of a neutrino with rotating spin in dense magnetized matter

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The problem of neutrino spin rotation in dense matter and in strong electromagnetic fields is solved in accordance with the basic principles of quantum mechanics. We obtain a complete system of wave functions for a massive Dirac neutrino with an anomalous magnetic moment which are the eigenfunctions of the kinetic momentum operator and have the form of nonspreading wave packets. These wave functions enable one to consider the states of neutrino with rotating spin as pure quantum states and can be used for calculating probabilities of various processes with the neutrino in the framework of the Furry picture.

INTRODUCTION

Neutrino physics is one of the most rapidly developing areas of high-energy physics. The fundamental experimental result was obtained in this field in recent years — neutrino oscillations were discovered [1]. From the theory of this phenomenon (see, for example, [2]), which is based on the ideas of Pontecorvo [3] and Maki, Nakagawa and Sakata [4], it follows that oscillations are possible only when the neutrino mass is nonzero. As a consequence of this circumstance, the possibility exists of neutrino spin rotation, i.e. of transitions between the active left-handed neutrino polarization state and the sterile right-handed state. As it is understood today, in contrast to flavor oscillations, this process can never be observed if the particle moves in vacuum. For the effect of spin rotation to exist some external influence leading to effective breaking of the Lorentz-symmetry of the theory is needed. For instance, dense matter or electromagnetic fields can serve as the physical reason for this phenomenon. In the latter case, nontrivial electromagnetic properties of the neutrino, in particular, the anomalous magnetic moment of a Dirac neutrino [5], can lead to rotation of its spin.

The conventional approach [6], [7] to the theory of neutrino spin rotation, based on solving the Cauchy problem for the Schrödinger-type equation with an effective Hamiltonian, was developed for the description of ultrarelativistic neutrinos (the particles that are observed in experiments now available). This approach is in fact quasiclassical and is equivalent to solving the Bargmann–Michel–Telegdi (BMT) equation [8] in the spinor representation (see [9]).

The aim of this work is to develop a consistent quantum theory of neutrino spin rotation to describe neutrinos with nonzero mass, including the low-energy region that may play a significant role in astrophysics [10]. Following papers [6], [7], we use the method of effective potential [11] that allows us to take into account collective influence of background particles on the propagation of neutrino.

To this end, we find a complete system of solutions of the Dirac-type equation that describe the action of both electromagnetic field and dense matter on neutrino dynamics. These solutions have the form of nonspreading wave packets. They enable one to consider the states of neutrino with rotating spin as pure quantum states and to evaluate probabilities of various processes with participation of a neutrino in the framework of the Furry picture [12].

The paper is organized as follows. In the first Section we present the model we are working with and briefly describe how the effective potentials are constructed. In Section II we summarize the main ideas of the method used for solving the Dirac equation. In Section III we introduce a kinetic momentum operator and justify the advantages of using its eigenvalues as quantum numbers, instead of the eigenvalues of the canonical momentum operator. Then we find both stationary and nonstationary wave functions. Here we also derive the dispersion law and calculate the group velocity for a neutrino in the dense magnetized matter. We demonstrate that the group velocity of a neutrino is independent of its spin orientation. In Section IV we find the explicit expressions for the kinetic momentum and spin projection

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operators. In Section V we use the obtained results for quasiclassical interpretation of neutrino spin behavior in dense magnetized matter. In the conclusive Section VI we compare our results with those obtained by other authors.

The units $\hbar = c = 1$ are used throughout the paper; metric tensor, $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$; totally antisymmetric Levi-Civita tensor, $e^{\mu\nu\rho\lambda} = +1$ if $\{\mu\nu\rho\lambda\}$ is an even permutation of $\{0, 1, 2, 3\}$, $e^{\mu\nu\rho\lambda} = -1$ if it is an odd permutation, $e^{\mu\nu\rho\lambda} = 0$ otherwise.

I. WAVE EQUATION FOR NEUTRINO IN DENSE MATTER AND IN ELECTROMAGNETIC FIELD

Let us first discuss the model we use for the description of the neutrino behavior in the matter and in the electromagnetic field. At present, there are no experimental data that might help to make the choice between the Dirac or the Majorana nature of a neutrino [13]. Therefore, in this paper we discuss spin rotation of a massive Dirac neutrino with anomalous magnetic moment propagating through dense magnetized matter.

As a first approximation, we can consider spin rotation independently of variations of the particle flavor, i.e. we can deal with the mass eigenstates of neutrino. It is quite obvious, due to the fact that the physical origins of neutrino flavor oscillations and its spin rotation are different. The Dirac–Pauli equation [14] serves as the basis of this approach in electromagnetic fields. Just in this way, the behavior of the neutrino spin in a constant homogeneous magnetic field was investigated in [15]. In order to consider the influence of matter on neutrino dynamics through weak interaction, the Dirac–Pauli equation should be modified.

As it was proposed in [11], [16], if matter density is high enough for considering weak interaction of neutrino with the background fermions as coherent, it is possible to describe neutrino interaction with matter by an effective potential. To get covariant realization of this idea, it is necessary to take into account that the term that describes direct interaction of neutrino with matter must depend on the currents

$$j_f^\mu = \{n_f u_f^0, n_f \mathbf{u}_f\}, \quad (1.1)$$

and on the polarizations

$$\lambda_f^\mu = \left\{ n_f (\boldsymbol{\zeta}_f \cdot \mathbf{u}_f), n_f \left(\boldsymbol{\zeta}_f + \frac{\mathbf{u}_f (\boldsymbol{\zeta}_f \cdot \mathbf{u}_f)}{1 + u_f^0} \right) \right\} \quad (1.2)$$

of the background fermions f [17–19]. It is significant that only j_f^μ and λ_f^μ characterize matter integrally. In formulas (1.1) and (1.2), n_f and $\boldsymbol{\zeta}_f$ ($0 \leq |\boldsymbol{\zeta}_f|^2 \leq 1$) are the number density and the mean value of the polarization vector of the background fermions f in the center-of-mass system of matter ($u_f^\mu = \{u_f^0, \mathbf{u}_f\}$ denotes the four-velocity of the reference frame), where the mean momentum of the fermions f is equal to zero.

In the framework of the standard model (for a more general case, see Appendix A), the interaction term can be derived directly from the model Lagrangian. To this end, it is necessary to write this Lagrangian in the low-energy effective form using the four-fermion interaction and replace the vector $\bar{\psi}_f(x)\gamma^\mu\psi_f(x)$ and axial vector $\bar{\psi}_f(x)\gamma^5\gamma^\mu\psi_f(x)$ currents of the background fermions by j_f^μ and λ_f^μ respectively. The explicit form of j_f^μ and λ_f^μ is obtained as a result of averaging of currents over the fermion statistical distribution function and depends on the type of the matter considered. In particular, the expression for the polarization λ_e^μ of electron gas in a magnetic field at finite temperature can be found in [7]. Note that, since the origin of the effective potential is forward elastic scattering of neutrino, it can also be calculated using field-theoretic methods [20].

As a result we come to the equation

$$\left(i\gamma^\mu \partial_\mu - \frac{1}{2}\gamma^\mu f_\mu (1 + \gamma^5) - \frac{i}{2}\mu_0 F^{\mu\nu} \sigma_{\mu\nu} - m \right) \Psi(x) = 0, \quad (1.3)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. The γ -matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ with γ^0 Hermitian, $\boldsymbol{\gamma}$ anti-Hermitian, and are related to the β and $\boldsymbol{\alpha}$ matrices through $\gamma^0 = \beta$, $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$; $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. The effective four-potential f^μ is a linear combination of currents and polarizations of the background fermions

$$f^\mu = \sum_f (\rho_f^{(1)} j_f^\mu + \rho_f^{(2)} \lambda_f^\mu). \quad (1.4)$$

Summation in (1.4) is carried out over all fermions f of matter. The expressions for the coefficients $\rho_f^{(1,2)}$ are determined as

$$\rho_f^{(1)} = \sqrt{2}G_F \left\{ I_{ev} + T_3^{(f)} - 2Q^{(f)} \sin^2 \theta_W \right\}, \quad \rho_f^{(2)} = -\sqrt{2}G_F \left\{ I_{ev} + T_3^{(f)} \right\}. \quad (1.5)$$

Here $Q^{(f)}$ is the electric charge of the fermion f ; $T_3^{(f)}$ is the third component of the weak isospin; G_F and θ_W are the Fermi constant and the Weinberg angle respectively; $I_{e\nu} = 1$ for the electron neutrino interaction with electrons, $I_{e\nu} = -1$ for interaction with positrons, otherwise $I_{e\nu} = 0$.

While equation (1.3) describes mass states of neutrino, potential (1.4) is flavor dependent, and in the general case, this leads to correlations between spin rotation and flavor oscillations. In order to construct a mathematically consistent approach to a description of neutrino spin rotation, we have to avoid the correlations. This is possible if we assume that the effective potential which describes the influence of dense matter on neutrino is the same for different flavors. In this case we can construct flavor states of the neutrino as the linear combination of its mass states with coefficients that are elements of the mixing matrix of the neutrino in vacuum.

The physical justification of this model can be made in the following way. In order to consider effective potentials as flavor independent, it is necessary to assume that the fraction of electrons in the matter is small, i.e. the electron number density n_e is approximately equal to zero. Calculations which are in a satisfactory agreement with the experimental data show that in the center of a neutron star the fraction of electrons does not exceed a few per cent [21]. Therefore the model approximation $n_e = 0$ is quite appropriate.

We restrict ourselves to considering equation (1.3) in the case of a constant homogeneous electromagnetic field and constant currents and polarizations of matter,

$$F^{\mu\nu} = \text{const}, \quad j_f^\mu = \text{const}, \quad \lambda_f^\mu = \text{const}, \quad (1.6)$$

as the first order approximation of the realistic background. Because of (1.6) additional conditions for $F^{\mu\nu}$ and f^μ may be obtained. The strengths of the electric and magnetic fields and average currents and polarizations of the background particles should obey the self-consistent system of equations including the Maxwell equations, the Lorentz equation

$$\dot{j}_f^\mu = \frac{e_f}{m_f} F^\mu{}_\nu j_f^\nu, \quad (1.7)$$

and the BMT quasiclassical spin evolution equation

$$\dot{\lambda}_f^\mu = \left[\frac{e_f}{m_f} F^\mu{}_\nu + 2\mu_f (g^{\mu\alpha} - u_f^\mu u_f^\alpha) F_{\alpha\nu} \right] \lambda_f^\nu. \quad (1.8)$$

Here the dot denotes differentiation with respect to the proper times τ_f of particles. From equations (1.7) and (1.8) we find that for charged particles the conditions $\dot{j}_f^\mu = 0$, $\dot{\lambda}_f^\mu = 0$ are equivalent to $F^\mu{}_\nu j_f^\nu = 0$, $F^\mu{}_\nu \lambda_f^\nu = 0$. However, it seems reasonable to assume that in dense matter velocities of the center-of-mass systems for all components are equal, thus for the neutral particles similar conditions should hold as well. In this way we obtain the restriction

$$F^{\mu\nu} f_\nu = 0. \quad (1.9)$$

It should be emphasized that condition (1.9) is the result of the fact that average currents and polarizations of particles of matter in an external field should satisfy the classical equations of motion. The physical meaning of this condition is discussed in more detail in Section V.

Note that equation (1.3) with constant coefficients appears in the standard model extension [22], [23] as well. In this case, $F^{\mu\nu}$ and f^μ describe vacuum condensates that break the Lorentz-symmetry of the theory.

II. FORMULATION OF THE PROBLEM

The Dirac equation, in particular equation (1.3), is a partial differential equation. Therefore, as it is known, its general solution is defined up to an arbitrary function. However, in quantum-mechanical applications we deal with the so-called complete integral which depends on a set of constants, i.e. quantum numbers. Those are the eigenvalues of some self-adjoint operators. For the classification of particle states it is necessary to introduce the complete set of the operators — integrals of motion. Note that in nonrelativistic quantum mechanics any self-adjoint operator may serve as an operator of observable, which is not the case in the relativistic mechanics, where only integrals of motion, commuting with the operator of the equation, can be treated as operators of observables [24]. The choice of the complete set of them is different in each particular case and should be adequate to the problem being solved.

In the case of the problem of spin evolution, operators of kinetic momentum components should be included in the complete set. It becomes obvious if we start with the following argument. The direction of particle polarization is well defined in its rest frame, and then one finds the polarization in the laboratory frame upon carrying out an appropriate

Lorentz transformation. This transformation is defined by the group velocity of the particle, in other words, by its kinetic momentum. This implies that it is not the canonical momentum but the kinetic momentum operator that defines the direction of particle propagation.

Now, there is only one problem that remains to be solved. The form of the kinetic momentum operator for a particle with spin propagating under the influence of external fields is not known beforehand. So we have to find a self-adjoint operator \mathbf{p}^μ with the eigenvalues q^μ , which satisfies the condition $q^2 = m^2$ and may be interpreted as the components of the particle kinetic momentum.

Let us discuss this issue in more detail. In the mathematical apparatus of quantum field theory, a particle is usually identified with an irreducible unitary representation of the Poincaré group [25]. The irreducible representations are characterized by two invariants of the group

$$P^2 \equiv P^\mu P_\mu = m^2, \quad (2.1)$$

$$W^2 \equiv W^\mu W_\mu = -m^2 s(s+1). \quad (2.2)$$

The translation generators P^μ are identified with the components of the particle momentum, and the Pauli–Lubanski–Bargmann vector

$$W^\mu = -\frac{1}{2} e^{\mu\nu\rho\lambda} M_{\nu\rho} P_\lambda, \quad (2.3)$$

where $M^{\mu\nu}$ are the Lorentz generators, characterizes the particle spin. The invariant m^2 is the particle mass squared and s is the value of its spin.

A space of unitary representation is defined by the condition called “the wave equation for a particle with mass m and spin s .” The wave equation for particles with spin $s = 1/2$ is the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \Psi_0(x) = 0. \quad (2.4)$$

In this case the realization of generators of the Poincaré group and the Pauli–Lubanski–Bargmann vector in the coordinate representation is

$$p^\mu = i\partial^\mu, \quad m^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) + \frac{i}{2} \sigma^{\mu\nu}, \quad w^\mu = \frac{i}{4} \gamma^5 \sigma^{\mu\nu} \partial_\nu. \quad (2.5)$$

These operators commute with the operator of the Dirac equation and can be identified with observables. They have a self-adjoint extension on the subsets of solutions of equation (2.4) with a fixed sign of the energy with regard to the standard scalar product,

$$(\Phi, \Psi) = \int d\mathbf{x} \Phi^\dagger(\mathbf{x}, t) \Psi(\mathbf{x}, t). \quad (2.6)$$

Three-dimensional spin projection operator $\mathbf{s}(p)$ is a set of coefficients $s_i(p)$ of the expansion of the vector w^μ in spacelike unit vectors $S_i^\mu(p)$ ($i = 1, 2, 3$):

$$s_i(p) = -\frac{1}{m} w_\mu S_i^\mu(p), \quad (2.7)$$

where

$$p_\mu S_i^\mu(p) = 0, \quad S_i^\mu(p) S_{\mu j}(p) = -\delta_{ij}. \quad (2.8)$$

Obviously,

$$[s_i(p), s_j(p)] = i e_{ijk} s_k(p). \quad (2.9)$$

The choice of these unit vectors is not unique, and it is possible to construct operators that determine the spin projection on any direction in an arbitrary Lorentz frame.

The above description of the particle characteristics cannot be directly used in the presence of external fields, where operators (2.5) are not necessarily integrals of motion. In this case the classification of particle states is usually realized by linear combinations of operators p^μ and w^μ with coefficients depending on coordinates [26]. Unfortunately, the physical meaning of these operators is usually not quite clear. Therefore, one should formulate a description of the particle motion in an external field in the same clear and detailed way as for a free particle.

This can be achieved basing on the following ideas. Since an irreducible representation of group is defined accurately up to an equivalence transformation, it is reasonable to state the problem of finding such realization of the Lie algebra of the Poincaré group for which the condition of irreducibility of the representation leads to wave equation describing a particle in a given external background. To solve this problem it is necessary to find a unitary operator $U(x, x_0)$ which converts solutions of the wave equation for a free particle (2.4) to solutions $\Psi(x)$ of (1.3):

$$U(x, x_0)\Psi_0(x) = \Psi(x). \quad (2.10)$$

Thus, $U(x, x_0)$ is an intertwining operator in the sense of Darboux [27]. This operator in our case should satisfy the equation

$$\left(i\gamma^\mu \partial_\mu - \frac{1}{2}\gamma^\mu f_\mu(1 + \gamma^5) - \frac{i}{2}\mu_0 F^{\mu\nu} \sigma_{\mu\nu} \right) U(x, x_0) - U(x, x_0) (i\gamma^\mu \partial_\mu) = 0. \quad (2.11)$$

Therefore, operators

$$\mathbf{p}^\mu = U(x, x_0)p^\mu U^{-1}(x, x_0), \quad \mathbf{m}^{\mu\nu} = U(x, x_0)m^{\mu\nu}U^{-1}(x, x_0) \quad (2.12)$$

commute with the operator of the wave equation. As a consequence, the Pauli–Lubanski–Bargmann vector \mathfrak{M}^μ and the components of the three-dimensional spin projection operator \mathfrak{S}_i can be constructed in the same way as in the case of a free particle:

$$\mathfrak{M}^\mu = -\frac{1}{2}e^{\mu\nu\rho\lambda}m_{\nu\rho}p_\lambda, \quad \mathfrak{S}_i = -\frac{1}{m}\mathfrak{M}_\mu S_i^\mu(\mathbf{p}). \quad (2.13)$$

The above statement may be reduced to the following: the wave function of a neutrino in dense matter can be derived with the help of a solution of the Dirac equation for a free particle and of some unitary evolution operator. A complete set of integrals of motion may be constructed with the help of operators (2.12). The physical meaning of eigenvalues of observables, i.e. quantum numbers, is clear enough then. However, in the general case, $U(x, x_0)$ is an integral operator and, as a consequence, operators (2.12) are also integral ones, so it is difficult to find $U(x, x_0)$ by direct calculations. That is why we are trying to find the evolution operator for equation (1.3) using the correspondence principle.

III. WAVE FUNCTIONS

When motion of a massive particle with spin in electromagnetic fields is described in the framework of the quasi-classical approach, we use two four-vectors, namely the four-velocity u^μ and the spin vector S^μ obeying the conditions

$$u^2 = 1, \quad S^2 = -1, \quad (uS) \equiv u^\mu S_\mu = 0. \quad (3.1)$$

These vectors are solutions of the Lorentz and BMT equations, respectively. As it was shown in [28], both four-vectors can be constructed with the help of one and the same evolution operator that acts on different initial values of these four-vectors, satisfying relations (3.1). In other words, the evolution operator for the BMT equation completely describes quasiclassical behavior of the particle.

As far as the neutrino is a neutral particle, the BMT equation for the spin vector S^μ of a neutrino moving with the four-velocity u^μ has the form

$$\dot{S}^\mu = 2\mu_0 (g^{\mu\alpha} - u^\mu u^\alpha) F_{\alpha\nu} S^\nu. \quad (3.2)$$

We can extend the BMT equation to include weak interaction between the neutrino and dense matter. It was shown phenomenologically in [29] that if effects of neutrino weak interactions are taken into account, the Lorentz-invariant generalization of the BMT equation is

$$\dot{S}^\mu = 2(g^{\mu\alpha} - u^\mu u^\alpha) (\mu_0 F_{\alpha\nu} + G_{\alpha\nu}) S^\nu, \quad (3.3)$$

where

$$G^{\mu\nu} = \frac{1}{2}e^{\mu\nu\rho\lambda}f_\rho u_\lambda. \quad (3.4)$$

We can get an analogous result by averaging the equations of motion for the Heisenberg operators (see [30]).

A remarkable feature of the BMT equation is its universality — at least in the zeroth order of the Planck constant, particles with an arbitrary spin may be described by the same BMT equation. However, a part of the information about the particle behavior gets lost due to distinctions in transformation properties of the states of the particles with different spins. This information may be restored in the following way. We can describe a particle using quasiclassical spin wave functions $\Psi(\tau)$ constructed in such a way that the vector current

$$j_V^\mu(\tau) = \bar{\Psi}(\tau)\gamma^\mu\Psi(\tau), \quad (3.5)$$

as well as the axial current

$$j_A^\mu(\tau) = \bar{\Psi}(\tau)\gamma^5\gamma^\mu\Psi(\tau) \quad (3.6)$$

built on their base, obey the BMT equation, and, as a consequence, may be interpreted as four-velocity u^μ and spin vector S^μ respectively.

We can introduce quasiclassical spin wave functions as follows [31], [32]. Let the Lorentz equation be solved, i.e. the dependence of the particle coordinates on proper time is found. Then the BMT equation transforms to an ordinary differential equation, whose resolvent determines a one-parameter subgroup of the Lorentz group. The quasiclassical spin wave function is a spin tensor, whose evolution is determined by the same one-parameter subgroup. It is easy to verify (see Appendix C) that for a neutral particle, represented by a Dirac bispinor, the equation that defines the evolution operator for the wave function $\Psi(\tau)$ takes the form

$$\dot{U}(\tau, \tau_0) = \frac{i}{m}\gamma^5\sigma_{\mu\nu}\varphi^\mu q^\nu U(\tau, \tau_0). \quad (3.7)$$

Here $\varphi^\mu = f^\mu/2 + H^{\mu\nu}q_\nu/m$, $H^{\mu\nu} = \mu_0 {}^*F^{\mu\nu}$, where ${}^*F^{\mu\nu} = -\frac{1}{2}e^{\mu\nu\rho\lambda}F_{\rho\lambda}$ is the dual electromagnetic field tensor and $q^\mu = mu^\mu$.

Let us look for a solution of equation (1.3) in the form

$$\Psi(x) = e^{-iK(x)}U(\tau(x), 0)\Psi_0(x), \quad (3.8)$$

where $U(\tau, \tau_0)$ is an evolution operator for the quasiclassical spin wave function, $e^{-iK(x)}$ is a phase factor, and $\Psi_0(x)$ is a solution of the Dirac equation for a free particle. Since we assume that $F^{\mu\nu}$ and f^μ are constants, the evolution operator can be expressed as a matrix exponential,

$$U(\tau, \tau_0) = \exp\left\{\frac{i}{m}\gamma^5\sigma_{\mu\nu}\varphi^\mu q^\nu(\tau - \tau_0)\right\}, \quad (3.9)$$

that depends on constant four-vector q^μ satisfying the condition $q^2 = m^2$.

Using ansatz (3.8) we should choose such a basis in the space of solutions of the Dirac equation for the free particle that the action of the evolution operator on each element of this basis be reduced to multiplication by one and the same matrix function depending on quantum numbers of the basis. It is obvious that in our case we must choose the plane waves

$$\Psi_0(x) = e^{-i(qx)}(1 - \zeta_0\gamma^5\gamma_\mu S_0^\mu(q))(\gamma^\mu q_\mu + m)\psi_0. \quad (3.10)$$

Here $S_0^\mu(q)$ determines the direction of polarization of the particle; $\zeta_0 = \pm 1$ is the spin projection on $S_0^\mu(q)$; ψ_0 is a constant four-component spinor. The wave function is normalized by the condition

$$\bar{\Psi}_0(x)\Psi_0(x) = m/q_0, \quad (3.11)$$

and four-vector q^μ is a kinetic momentum of the particle. Though the explicit form of a kinetic momentum operator for a particle with spin interacting with dense matter and electromagnetic field is not known beforehand, the correspondence principle allows us to construct solutions characterized by its eigenvalues.

Naturally, as far as equation (1.3) is invariant under translations the canonical momentum operator $p^\mu = i\partial^\mu$ is an integral of motion for this equation, too. However, the commonly adopted choice of eigenvalues of this operator as quantum numbers is not satisfactory if we prefer spin projection operators with clear physical meaning. The directions of canonical and kinetic momenta are different in the general case (see [23]) and, as it was already mentioned in Section II, projection of the spin is well defined in the rest frame of the particle where its kinetic momentum is equal to zero: $\mathbf{q} = 0$. That is why in the construction of spin projection operators (see (2.7)) it is necessary to select unit vectors orthogonal to four-vector q^μ .

Let us find the proper time $\tau(x)$ and the phase factor $K(x)$. Substitution of (3.8) into (1.3) gives

$$\left\{ \gamma^\mu q_\mu + \gamma^\mu \partial_\mu K(x) - \frac{1}{2} \gamma^\mu f_\mu (1 + \gamma^5) + \frac{1}{m} \gamma^5 \sigma_{\mu\nu} \varphi^\mu q^\nu \gamma^\alpha N_\alpha - \frac{i}{2} \mu_0 F^{\mu\nu} \sigma_{\mu\nu} - m \right\} e^{-iK(x)} U(\tau(x)) \Psi_0(x) = 0,$$

where $N^\mu = \partial^\mu \tau$. Since the matrix $U(\tau(x))$ is nondegenerate and the commutator $[\gamma^\mu q_\mu, U(\tau(x))]$ is zero, we get

$$\gamma^\mu \partial_\mu K(x) - \frac{1}{2} \gamma^\mu f_\mu (1 + \gamma^5) + \frac{1}{m} \gamma^5 \sigma_{\mu\nu} \varphi^\mu q^\nu \gamma^\alpha N_\alpha - \frac{i}{2} \mu_0 F^{\mu\nu} \sigma_{\mu\nu} = 0.$$

To solve this equation for N^μ and $K(x)$, we should set the coefficients at the linearly independent elements of the algebra of the Dirac matrices equal to zero. Thus

$$\partial^\mu K(x) = f^\mu / 2, \quad (3.12)$$

and the system that defines the vector N^μ is

$$\varphi^\mu (m - (Nq)) + (N\varphi) q^\mu = 0, \quad \mu_0 F^{\mu\alpha} q_\alpha = -e^{\mu\nu\rho\lambda} N_\nu \varphi_\rho q_\lambda, \quad (3.13)$$

The system is consistent provided that

$$q_\mu F^{\mu\nu} \varphi_\nu = 0. \quad (3.14)$$

Equation (3.14) must be held for an arbitrary q^μ , so we have

$$*F^{\mu\alpha} F_{\alpha\nu} \equiv -\frac{1}{4} \delta_\nu^\mu *F^{\alpha\beta} F_{\alpha\beta} = 0, \quad (3.15)$$

$$F^{\mu\nu} f_\nu = 0. \quad (3.16)$$

Any antisymmetric tensor has an eigenvector corresponding to zero eigenvalue if and only if its second invariant $I_2 = \frac{1}{4} *F^{\mu\nu} F_{\mu\nu}$ is equal to zero. That is why conditions (3.15) and (3.16) are not independent and (3.15) is a consequence of (3.16). Comparing it with (1.9), we see that the condition obtained basing on the physical reasons alone and condition (3.16) totally coincide. This ensures quasiclassical behavior of the particle in the background medium and allows obtaining the solution of equation (1.3) in the form of (3.8).

Using an orthogonal basis in the Minkowski space,

$$n_0^\mu = q^\mu / m, \quad n_1^\mu = \frac{H^{\mu\nu} q_\nu}{\sqrt{\mathcal{N}}}, \quad n_2^\mu = \frac{F^{\mu\nu} q_\nu}{\sqrt{\tilde{\mathcal{N}}}}, \quad n_3^\mu = \frac{m^2 H^{\mu\nu} H_{\nu\alpha} q^\alpha - q^\mu \mathcal{N}}{m \sqrt{\mathcal{N} \tilde{\mathcal{N}}}}, \quad (3.17)$$

where $\mathcal{N} = q_\mu H^{\mu\rho} H_{\rho\nu} q^\nu$, $\tilde{\mathcal{N}} = \mu_0^2 q_\mu F^{\mu\rho} F_{\rho\nu} q^\nu$, and relations (B3), (B4) from Appendix B, we find

$$N^\mu = -q^\mu \frac{m(f\varphi)}{2((\varphi q)^2 - m^2 \varphi^2)} + f^\mu \frac{m}{2(\varphi q)} + \varphi^\mu \frac{m^3(f\varphi)}{2(\varphi q)((\varphi q)^2 - m^2 \varphi^2)}. \quad (3.18)$$

According to the fact that f^μ and N^μ are constant values we obtain the proper time

$$\tau = (Nx), \quad (3.19)$$

and the phase factor

$$K(x) = (fx)/2, \quad (3.20)$$

which determines an energy shift of the neutrino in matter.

Now we can derive the expression for the wave function:

$$\Psi_{q\zeta_0}(x) = \frac{1}{2} \sum_{\zeta=\pm 1} e^{-i(P_\zeta x)} (1 - \zeta \gamma^5 \gamma_\mu S_{\text{tp}}^\mu(q)) (1 - \zeta_0 \gamma^5 \gamma_\mu S_0^\mu(q)) (\gamma^\mu q_\mu + m) \psi_0, \quad (3.21)$$

where

$$P_\zeta^\mu = q^\mu + f^\mu/2 - \zeta N^\mu \sqrt{(\varphi q)^2 - \varphi^2 m^2}/m = q^\mu \left(1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right) + \frac{1}{2} f^\mu \left(1 - \frac{\zeta \sqrt{(\varphi q)^2 - m^2 \varphi^2}}{(\varphi q)} \right) - \varphi^\mu \frac{\zeta (f\varphi) m^2}{2(\varphi q) \sqrt{(\varphi q)^2 - m^2 \varphi^2}}, \quad (3.22)$$

$$S_{\text{tp}}^\mu(q) = \frac{q^\mu(\varphi q)/m - \varphi^\mu m}{\sqrt{(\varphi q)^2 - \varphi^2 m^2}}. \quad (3.23)$$

System (3.21) represents the complete system of solutions of equation (1.3) characterized by kinetic momentum of the particle q^μ and the quantum number $\zeta_0 = \pm 1$ which can be interpreted as the neutrino spin projection on the direction $S_0^\mu(q)$ at $\tau = (Nx) = 0$.

System (3.21) is nonstationary in the general case. The solutions are stationary only when the initial polarization vector $S_0^\mu(q)$ is equal to the vector of the total polarization $S_{\text{tp}}^\mu(q)$ [33], $S_0^\mu(q) = S_{\text{tp}}^\mu(q)$. In this case the wave functions are eigenfunctions of the spin projection operator $\mathfrak{S}_{\text{tp}} = -\gamma^5 \gamma_\mu S_{\text{tp}}^\mu(q)$ with eigenvalues $\zeta = \pm 1$, and of the canonical momentum operator $p^\mu = i\partial^\mu$ with eigenvalues P_ζ^μ . The orthonormal system of the stationary solutions, the basis of solutions of equation (1.3), can be written as (see Appendix D)

$$\Psi_{q\zeta}(x) = e^{-i(P_\zeta x)} \sqrt{|J_\zeta(q)|} (1 - \zeta \gamma^5 \gamma_\mu S_{\text{tp}}^\mu(q)) (\gamma^\mu q_\mu + m) \psi_0, \quad (3.24)$$

where $J_\zeta(q)$ is the transition Jacobian between the variables q^μ and P_ζ^μ

$$J_\zeta(q) = \det(M_{ij}) = \det \left[\frac{\partial P_\zeta^i}{\partial q^j} + \frac{\partial P_\zeta^i}{\partial q^0} \frac{\partial q^0}{\partial q^j} \right]. \quad (3.25)$$

With the help of relations from Appendix B, the explicit form of the matrix M_{ij} may be written as

$$M_{ij} = \delta_{ij} \left(1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right) + \zeta \left(q_i - \varphi_i \frac{m^2}{(\varphi q)} \right) \left(\varphi_j - q_j \frac{\varphi^0}{q^0} \right) \frac{m^2 (f\varphi)^2 + f^2 (\varphi q)^2 - m^2 f^2 \varphi^2}{4(\varphi q) ((\varphi q)^2 - m^2 \varphi^2)^{3/2}}.$$

One can easily derive the following equality for arbitrary vectors \mathbf{g} and \mathbf{h}

$$\det(\delta_{ij} + g_i h_j) = 1 + (\mathbf{g} \cdot \mathbf{h}).$$

So we have

$$J_\zeta(q) = \left(1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right)^2 \left(1 + \zeta \frac{f_\mu H^{\mu\nu} q_\nu / (2m) - 2\mu_0^2 I_1}{\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right). \quad (3.26)$$

Here $I_1 = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ is the first invariant of the tensor $F^{\mu\nu}$. Note that to obtain a complete system of solutions for the antineutrino, it is necessary to change the sign of the kinetic momentum q^μ .

The dispersion law for the neutrino in dense magnetized matter is different from the one for the free particle and can be written as (see relations (B7), (B8) and (B9) in Appendix B)

$$\tilde{P}^2 = m^2 - f^2/4 - 2\mu_0^2 I_1 - 2\zeta \Delta \sqrt{(\tilde{P}\tilde{\Phi})^2 - \tilde{\Phi}^2 m^2}, \quad (3.27)$$

where

$$\tilde{P}^\mu = P_\zeta^\mu - f^\mu/2, \quad \tilde{\Phi}^\mu = f^\mu/2 + H^{\mu\nu} \tilde{P}_\nu/m, \quad \Delta = \text{sign} \left(1 + \zeta \frac{f_\mu H^{\mu\nu} q_\nu / (2m) - 2\mu_0^2 I_1}{\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right) = \text{sign} \left(1 + \frac{f_\mu H^{\mu\nu} \tilde{P}_\nu / m - 4\mu_0^2 I_1}{\tilde{P}^2 - m^2 + f^2/4 + 2\mu_0^2 I_1 - (\tilde{\Phi} f)} \right). \quad (3.28)$$

The appearance of the factor Δ in equation (3.27) is a consequence of the fact that ζ is projection of the particle spin on the direction defined by the kinetic momentum instead of the canonical one.

In spite of the modification of the dispersion law described above, we see that the neutrino moving through dense matter and electromagnetic field may still behave as a free particle, i.e. its group velocity

$$\mathbf{v}_{\text{gr}} = \frac{\partial P_{\zeta}^0}{\partial \mathbf{P}_{\zeta}} = \frac{\mathbf{q}}{q^0} \quad (3.29)$$

is the same for both polarization states of the particle. However, in interactions with other particles some channels of reactions which are closed for a free neutrino can be opened due to the modification of the dispersion law (see, for example, [19], [36], [37]).

Let us discuss now properties of nonstationary solutions in more detail. Solution (3.21) is a plane-wave solution of equation (1.3), describing a pure quantum-mechanical state of a neutral particle with a nonconserved spin projection on the fixed space axis. Solutions (3.21) do not form an orthogonal basis. However, the considered system is not overcomplete, since the spectrum of the spin projection operator is finite. So the system can be easily orthogonalized. Generalization of the basis (3.24) is

$$\tilde{\Psi}_{q\zeta_0}(x) = \frac{1}{2} \sum_{\zeta=\pm 1} e^{-i(P_{\zeta}x)} \sqrt{|J_{\zeta}(q)|} (1 - \zeta\gamma^5\gamma_{\mu}S_{\text{tp}}^{\mu}(q))(1 - \zeta_0\gamma^5\gamma_{\mu}S_0^{\mu}(q))(\gamma^{\mu}q_{\mu} + m)\psi_0. \quad (3.30)$$

Thus we have just established that the unitary intertwining operator (2.10) in our case is the Fourier integral operator [38] and it acts on elements of the plane-wave basis of solutions of the free particle Dirac equation (3.10) in the following way:

$$\tilde{\Psi}_{q\zeta_0}(x) = U(x, x_0)\Psi_0(x) = \frac{1}{2} \sum_{\zeta=\pm 1} e^{-i((P_{\zeta}-q)x)} \sqrt{|J_{\zeta}(q)|} (1 - \zeta\gamma^5\gamma_{\mu}S_{\text{tp}}^{\mu}(q))\Psi_0(x). \quad (3.31)$$

Action of the inverse operator is defined by the formula

$$\Psi_0(x) = U^{-1}(x, x_0)\tilde{\Psi}_{q\zeta_0}(x) = \frac{1}{2} \sum_{\zeta=\pm 1} e^{i((P_{\zeta}-q)x)} \frac{1}{\sqrt{|J_{\zeta}(q)|}} (1 - \zeta\gamma^5\gamma_{\mu}S_{\text{tp}}^{\mu}(q))\tilde{\Psi}_{q\zeta_0}(x). \quad (3.32)$$

Since the intertwining operator is defined on the elements of the basis, its action on an arbitrary solution is defined as well. Hence, the explicit form of this operator as a function of coordinates and differential operators can be easily obtained.

Note that an attempt to construct intertwining operator for equation (1.3) was undertaken in [39] for the case where only parameter f^{μ} is nontrivial. The result of the action of the operator suggested in [39] on plane-wave solutions of the Dirac equation for a free particle coincides with solutions obtained in our previous work [19] for this particular case, and which can be derived from (3.21), if one sets $F^{\mu\nu} = 0$. We should emphasize that solutions (3.21) do not form an orthogonal basis. As a consequence of this fact, the intertwining operator suggested in [39] is not unitary with regard to the standard scalar product (2.6).

The case of a massless neutrino is quite special. Equation (3.7) obviously does not hold in the limit $m \rightarrow 0$ because massless particle helicity is conserved, while the BMT equation describes spin rotation. To find wave functions of massless neutrino, one must take into account that from the mathematical point of view, the small group of representation of the Poincaré group for a massive particle differ from that for a massless particle (see, for example, [25]). The small group for a massive particle is the rotation group of three-dimensional Euclidian space $O_+(3)$, and nontrivial magnetic and electric moments are allowed. In contrast to that, the small group for a massless particle is the movement group of Euclidian plane $E(2)$, which outlaws nonzero magnetic moment as well as electric one or other nontrivial coefficients μ_n and ε'_n in equation (A2). Therefore, one has to put $\mu_0 = 0$ in equation (1.3). Then instead of (3.24), we obtain

$$\Psi_{q\zeta}(x)|_{m=0} = e^{-i(qx)(1+\zeta\eta)} e^{-i(fx)(1-\zeta)/2} |1 + \zeta\eta| (1 - \zeta\gamma^5)\gamma^{\mu}q_{\mu}\psi_0, \quad (3.33)$$

for the neutrino, and

$$\Psi_{q\zeta}(x)|_{m=0} = e^{i(qx)(1+\zeta\eta)} e^{-i(fx)(1+\zeta)/2} |1 + \zeta\eta| (1 + \zeta\gamma^5)\gamma^{\mu}q_{\mu}\psi_0, \quad (3.34)$$

for the antineutrino. In these equations $\eta = \frac{f^2}{2(fq)}$ and $q^2 = 0$.

IV. OPERATORS OF OBSERVABLES

Let us find the explicit forms of the kinetic momentum operators \mathbf{p}^μ and the spin projection operator \mathfrak{S}_{tp} in the coordinate representation. For this purpose we might exploit formulas (2.12); however, we follow a simpler way.

The obtained solutions (3.24) are classified by eigenvalues of the operators \mathbf{p}^μ and \mathfrak{S}_{tp} , so

$$\mathbf{p}^\mu \Psi_{q\zeta}(x) = q^\mu \Psi_{q\zeta}(x), \quad \mathfrak{S}_{\text{tp}} \Psi_{q\zeta}(x) = \zeta \Psi_{q\zeta}(x). \quad (4.1)$$

Since solutions (3.24) are also eigenfunctions of the canonical momentum operator $p^\mu = i\partial^\mu$ with eigenvalues P_ζ^μ , we have

$$p^\mu \Psi_{q\zeta}(x) = P_\zeta^\mu \Psi_{q\zeta}(x). \quad (4.2)$$

Now we should express eigenvalues of the kinetic momentum operator q^μ in terms of eigenvalues of the canonical momentum operator P_ζ^μ . From (3.22) we have

$$q^\mu = \left[\tilde{P}^\mu + f^\mu \frac{\zeta \sqrt{(\varphi q)^2 - m^2 \varphi^2}}{2(\varphi q)} + \varphi^\mu \frac{\zeta (f\varphi) m^2}{2(\varphi q) \sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right] \times \left[1 + \zeta \frac{(f\varphi)}{2\sqrt{(\varphi q)^2 - m^2 \varphi^2}} \right]^{-1}. \quad (4.3)$$

With the help of relations from Appendix B this can be rewritten as

$$q^\mu = \tilde{P}^\mu + \frac{\tilde{P}^\mu (\tilde{\Phi} f) - f^\mu (f \tilde{P})/2 - 2m H^{\mu\nu} \tilde{\Phi}_\nu}{\tilde{P}^2 - m^2 + f^2/4 + 2\mu_0^2 I_1 - (\tilde{\Phi} f)}. \quad (4.4)$$

The vector of total polarization in terms of the new variable is

$$S_{\text{tp}}^\mu(q) = \Delta \frac{q^\mu (\tilde{\Phi} \tilde{P})/m - \tilde{\Phi}^\mu m}{\sqrt{(\tilde{\Phi} \tilde{P})^2 - m^2 \tilde{\Phi}^2}}. \quad (4.5)$$

Here \tilde{P} , $\tilde{\Phi}$ and Δ are given by (3.28).

Because of (4.2) we can interpret P_ζ^μ as a result of action of operator $p^\mu = i\partial^\mu$ on the wave function. So by changing $\tilde{P}^\mu \Rightarrow p^\mu - f^\mu/2$ and $\tilde{\Phi}^\mu \Rightarrow f^\mu/2 + H^{\mu\nu}(p_\nu - f_\nu/2)/m$ in formulas (4.4), (4.5), we obtain kinetic momentum operator \mathbf{p}^μ and spin projection operator $\mathfrak{S}_{\text{tp}} = -\gamma^5 \gamma_\mu S_{\text{tp}}^\mu(q)$ in the explicit form. These operators are pseudodifferential ones [38] and are determined on the solutions of equation (1.3) with fixed mass m .

To extend the domain of the definition of constructed operators, we need to replace mass m in (4.4) and (4.5) by the matrix operator from equation (1.3):

$$m = \gamma^\mu p_\mu - \frac{1}{2} \gamma^\mu f_\mu (1 + \gamma^5) - \frac{i}{2} \mu_0 F^{\mu\nu} \sigma_{\mu\nu}, \quad (4.6)$$

$$m^2 = (p - f/2)^2 - f^2/4 + 2\mu_0^2 I_1 + \gamma^5 \sigma^{\mu\nu} f_\mu p_\nu + H^{\mu\nu} \gamma_\mu f_\nu (1 + \gamma^5) + 2\gamma^5 H^{\mu\nu} \gamma_\mu p_\nu.$$

In this way we obtain the covariant form for \mathbf{p}^μ and \mathfrak{S}_{tp} .

Unfortunately, the result of this substitution cannot be written as a compact formula, so we do not present it here. However, even if we do not know a covariant form of the operator \mathbf{p}^μ , we may conclude that on the solutions of equation (1.3) the relations,

$$\mathbf{p}^2 = m^2, \quad \gamma^\mu \mathbf{p}_\mu = m, \quad (4.7)$$

should hold. The first equality in (4.7) is obvious; to proof the second one, recall that according to (2.12)

$$m = U(x, x_0) \gamma^\mu p_\mu U^{-1}(x, x_0) = U(x, x_0) \gamma^\mu U^{-1}(x, x_0) \mathbf{p}_\mu = (\gamma^\mu + \Gamma^\mu(q)) \mathbf{p}_\mu. \quad (4.8)$$

From formulas (3.31) and (3.32) it follows that $\Gamma^\mu(q) \sim S_{\text{tp}}^\mu(q)$, and as a consequence $\Gamma^\mu(q) \mathbf{p}_\mu = 0$. So equation (1.3) may be represented as

$$(\gamma^\mu \mathbf{p}_\mu - m) \Psi(x) = 0. \quad (4.9)$$

Consider now special cases where the presented technique looks quite clear. Discuss the influence on the neutrino dynamics of the electromagnetic field alone, i.e. assume that $f^\mu = 0$. In this case eigenvalues of the kinetic momentum operator q^μ would be expressible in terms of eigenvalues of the canonical momentum operator P_ζ^μ in the following way:

$$q^\mu = P_\zeta^\mu - \frac{2H^{\mu\alpha}H_{\alpha\nu}P_\zeta^\nu}{P_\zeta^2 - m^2 + 2\mu_0^2 I_1}. \quad (4.10)$$

Then the covariant form of the kinetic momentum operator is

$$\mathbf{p}^\mu = p^\mu + \frac{H^{\mu\alpha}H_{\alpha\nu}p^\nu}{\sqrt{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}} \tilde{\mathfrak{S}}_{\text{tp}}, \quad (4.11)$$

and the spin projection operator \mathfrak{S}_{tp} is defined by the formula

$$\mathfrak{S}_{\text{tp}} = \text{sign} \left(1 + \frac{2\mu_0^2 I_1}{\sqrt{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}} \tilde{\mathfrak{S}}_{\text{tp}} \right) \tilde{\mathfrak{S}}_{\text{tp}}. \quad (4.12)$$

Here

$$\tilde{\mathfrak{S}}_{\text{tp}} = \frac{\gamma^5 \gamma_\mu H^{\mu\nu} p_\nu}{\sqrt{p^\beta H_{\beta\alpha}H^{\alpha\rho}p_\rho}}. \quad (4.13)$$

Thus operator \mathfrak{S}_{tp} has a simple physical meaning. It characterizes a particle spin projection on the direction of the magnetic field in the rest frame of the particle.

Vice versa, when the electromagnetic field is absent, but f^μ is nontrivial, we have

$$q^\mu = P_\zeta^\mu - \frac{1}{2}f^\mu + \frac{P_\zeta^\mu f^2 - f^\mu (f P_\zeta)}{2(P_\zeta^2 - (P_\zeta f) - m^2)}, \quad (4.14)$$

and the covariant forms of the kinetic momentum and spin projection operators are

$$\mathbf{p}^\mu = p^\mu - \frac{f^\mu}{2} - \gamma^5 \frac{p^\mu f^2 - f^\mu (f p)}{2((p f)^2 - p^2 f^2)} \sigma^{\alpha\nu} f_\alpha p_\nu, \quad (4.15)$$

$$\mathfrak{S}_{\text{tp}} = \frac{\gamma^5 \sigma^{\mu\nu} f_\mu p_\nu}{\sqrt{(p f)^2 - p^2 f^2}}. \quad (4.16)$$

Note that if the matter is at rest and nonpolarized ($\mathbf{f} = 0$), then

$$\mathfrak{S}_{\text{tp}} = \text{sign}(f^0) \frac{(\boldsymbol{\Sigma} \cdot \mathbf{p})}{|\mathbf{p}|}, \quad (4.17)$$

in other words \mathfrak{S}_{tp} is equal to the standard helicity operator up to the sign.

We can find now spin projection operators for nonstationary wave functions (3.21) and (3.30). For this purpose, we introduce operators \mathfrak{S}_\pm that act on the elements of system (3.24) as follows:

$$\mathfrak{S}_+ \Psi_{q\zeta}(x) = \frac{(1-\zeta)}{2} \Psi_{q(-\zeta)}(x), \quad \mathfrak{S}_- \Psi_{q\zeta}(x) = \frac{(1+\zeta)}{2} \Psi_{q(-\zeta)}(x). \quad (4.18)$$

Then operators $\mathfrak{S}_1 = \frac{1}{2}(\mathfrak{S}_+ + \mathfrak{S}_-)$, $\mathfrak{S}_2 = \frac{1}{2i}(\mathfrak{S}_+ - \mathfrak{S}_-)$ and $\mathfrak{S}_3 = \frac{1}{2}\mathfrak{S}_{\text{tp}}$ correspond to elements of the Lie algebra of the $SU(2)$ group. Commutation relations for these operators are

$$[\mathfrak{S}_i, \mathfrak{S}_j] = i e_{ijk} \mathfrak{S}_k. \quad (4.19)$$

To determine the explicit realization of operators \mathfrak{S}_\pm on eigenfunctions of operator \mathbf{p}^μ let us choose the basis $S_i^\mu(q)$ (see (2.8)) in the form $S_{\text{tp}}^\mu(q), S_1^\mu(q), S_2^\mu(q)$. Here $S_{\text{tp}}^\mu(q)$ is defined by relation (3.23), the other spacelike unit vectors are

$$S_1^\mu(q) = \frac{S_0^\mu(q) + S_{\text{tp}}^\mu(q)(S_0(q)S_{\text{tp}}(q))}{\sqrt{1 - (S_0(q)S_{\text{tp}}(q))^2}}, \quad S_2^\mu(q) = \frac{e^{\mu\nu\rho\lambda} q_\nu S_{0\rho}(q) S_{\text{tp}\lambda}(q)}{m \sqrt{1 - (S_0(q)S_{\text{tp}}(q))^2}}. \quad (4.20)$$

As a result we have

$$\mathfrak{S}_\pm = -\frac{1}{2} \frac{\sqrt{|J_{\zeta=\pm 1}(q)|}}{\sqrt{|J_{\zeta=\mp 1}(q)|}} e^{\pm 2i\theta} \gamma^5 \gamma_\mu (S_1^\mu(q) \pm iS_2^\mu(q)), \quad (4.21)$$

where

$$\theta = (Nx) \sqrt{(\varphi q)^2 - \varphi^2 m^2} / m. \quad (4.22)$$

Operators \mathfrak{S}_{tp} and \mathfrak{S}_\pm are integrals of motion. So the spin projection operator $\tilde{\mathfrak{S}}_0$ that has eigenfunctions (3.30) and eigenvalues $\zeta_0 = \pm 1$ is a linear combination of these operators. We can rewrite wave functions (3.30) in the form

$$\tilde{\Psi}_{q\zeta_0}(x) = \sqrt{1 - \zeta_0(S_0(q)S_{\text{tp}}(q))} \frac{\Psi_{q\zeta=1}(x)}{\sqrt{2}} + \zeta_0 \sqrt{1 + \zeta_0(S_0(q)S_{\text{tp}}(q))} \frac{\Psi_{q\zeta=-1}(x)}{\sqrt{2}}. \quad (4.23)$$

So it is obvious that

$$\tilde{\mathfrak{S}}_0 = -(S_0(q)S_{\text{tp}}(q))\mathfrak{S}_{\text{tp}} + \sqrt{1 - (S_0(q)S_{\text{tp}}(q))^2} [\mathfrak{S}_+ + \mathfrak{S}_-]. \quad (4.24)$$

Similarly to (4.24), one can construct the integral of motion \mathfrak{S}_0 with eigenfunctions (3.21) and eigenvalues $\zeta_0 = \pm 1$. Since

$$\Psi_{q\zeta_0}(x) = \sqrt{1 - \zeta_0(S_0(q)S_{\text{tp}}(q))} \frac{\Psi_{q\zeta=1}(x)}{\sqrt{2|J_{\zeta=+1}(q)|}} + \zeta_0 \sqrt{1 + \zeta_0(S_0(q)S_{\text{tp}}(q))} \frac{\Psi_{q\zeta=-1}(x)}{\sqrt{2|J_{\zeta=-1}(q)|}}, \quad (4.25)$$

we have

$$\mathfrak{S}_0 = -(S_0(q)S_{\text{tp}}(q))\mathfrak{S}_{\text{tp}} + \sqrt{1 - (S_0(q)S_{\text{tp}}(q))^2} \left[\frac{\sqrt{|J_{\zeta=-1}(q)|}}{\sqrt{|J_{\zeta=+1}(q)|}} \mathfrak{S}_+ + \frac{\sqrt{|J_{\zeta=+1}(q)|}}{\sqrt{|J_{\zeta=-1}(q)|}} \mathfrak{S}_- \right]. \quad (4.26)$$

Or, in the other form,

$$\mathfrak{S}_0 = \gamma^5 \gamma_\mu \left\{ (S_0(q)S_{\text{tp}}(q))S_{\text{tp}}^\mu(q) - \sqrt{1 - (S_0(q)S_{\text{tp}}(q))^2} [\cos 2\theta S_1^\mu(q) - \sin 2\theta S_2^\mu(q)] \right\}. \quad (4.27)$$

Note that operator (4.26) is not a self-adjoint operator with respect to the standard scalar product (2.6). It seems quite natural, since wave functions (3.21) do not form an orthogonal system. However, the system of wave functions is orthonormalized to the condition “one particle in the unit volume.” In this sense wave functions (3.21) minimize the uncertainty relation which is due to (4.19)

$$\langle (\mathfrak{S}_1 - \langle \mathfrak{S}_1 \rangle)^2 \rangle \langle (\mathfrak{S}_2 - \langle \mathfrak{S}_2 \rangle)^2 \rangle = \frac{1}{4} \langle \mathfrak{S}_3 \rangle^2. \quad (4.28)$$

Therefore, these wave functions describe spin-coherent states of the neutrino (on properties of coherent states see, for example, [40]). The given system of spin-coherent states is parametrized by four-vector $S_0^\mu(q)$.

V. QUASICLASSICAL INTERPRETATION

Let us discuss in more detail the physical meaning of spin-coherent states of the neutrino. For this purpose we construct vector and axial currents with the help of (3.21). The vector current is

$$j_V^\mu = \bar{\Psi}(x) \gamma^\mu \Psi(x) = q^\mu / q^0, \quad (5.1)$$

and we see that (3.21) describes neutrino propagation with the constant velocity $\mathbf{v}_{\text{gr}} = \mathbf{q}/q^0$. The axial current is

$$j_A^\mu = \bar{\Psi}(x) \gamma^5 \gamma^\mu \Psi(x) = \zeta_0 \frac{m}{q^0} S^\mu. \quad (5.2)$$

Here

$$S^\mu = -S_{\text{tp}}^\mu(q)(S_0(q)S_{\text{tp}}(q)) + [S_0^\mu(q) + S_{\text{tp}}^\mu(q)(S_0(q)S_{\text{tp}}(q))] \cos 2\theta - \frac{1}{m} e^{\mu\nu\rho\lambda} q_\nu S_{0\rho}(q) S_{\text{tp}\lambda}(q) \sin 2\theta, \quad (5.3)$$

where θ is determined by (4.22). As expected (see (3.5) and (3.6)), vector and axial currents coincide with the solutions of the BMT equation, if the proper time is defined as in (3.19).

The three-vector of spin $\boldsymbol{\zeta}$ can be expressed in terms of the four-vector S^μ components as

$$\boldsymbol{\zeta} = \mathbf{S} - \frac{\mathbf{q}S^0}{q^0 + m}. \quad (5.4)$$

Then for $\boldsymbol{\zeta}$ we have

$$\boldsymbol{\zeta} = \boldsymbol{\zeta}_{\text{tp}}(\boldsymbol{\zeta}_0 \cdot \boldsymbol{\zeta}_{\text{tp}}) + [\boldsymbol{\zeta}_0 - \boldsymbol{\zeta}_{\text{tp}}(\boldsymbol{\zeta}_0 \cdot \boldsymbol{\zeta}_{\text{tp}})] \cos 2\theta - [\boldsymbol{\zeta}_{\text{tp}} \times \boldsymbol{\zeta}_0] \sin 2\theta. \quad (5.5)$$

Expression (5.5) has a simple quasiclassical interpretation.

The antisymmetric tensor $G^{\mu\nu}$ (see equation (3.4)) can be written in the standard form

$$G^{\mu\nu} = (\mu_0 \mathbf{P}, \mu_0 \mathbf{M}), \quad (5.6)$$

where

$$\mathbf{M} = (f^0 \mathbf{q} - q^0 \mathbf{f})/(2\mu_0 m), \quad \mathbf{P} = -[\mathbf{q} \times \mathbf{f}]/(2\mu_0 m). \quad (5.7)$$

Vectors \mathbf{P} and \mathbf{M} are analogous to the polarization and the magnetization vectors of medium. Note that the substitution $F^{\mu\nu} \Rightarrow F^{\mu\nu} + G^{\mu\nu}/\mu_0$ implies that the magnetic \mathbf{H} and electric \mathbf{D} fields are shifted by the vectors \mathbf{M} and \mathbf{P} , respectively (we use here notation common in electrodynamics of continuous media [41])

$$\mathbf{H} \Rightarrow \mathbf{B} = \mathbf{H} + \mathbf{M}, \quad \mathbf{D} \Rightarrow \mathbf{E} = \mathbf{D} - \mathbf{P}. \quad (5.8)$$

Thus restriction (1.9) in the explicit form is

$$(\mathbf{E} \cdot \mathbf{f}) = 0, \quad \mathbf{E}f^0 - [\mathbf{B} \times \mathbf{f}] = 0. \quad (5.9)$$

This means that the Lorentz force and moment of force acting on matter are equal to zero, i.e. matter is at equilibrium state within the accuracy of our consideration. In particular, the vector of polarization of matter is parallel to the magnetic field if the matter is at rest.

In the rest frame of the particle, equation (3.3) can be written in the form

$$\dot{\boldsymbol{\zeta}} = 2\mu_0 [\boldsymbol{\zeta} \times \mathbf{B}_0], \quad (5.10)$$

where the spin vector $\boldsymbol{\zeta}$ is related to four-vector S^μ by equation (5.4) and the value \mathbf{B}_0 is the effective magnetic field in the neutrino rest frame. This field can be expressed in terms of quantities determined in the laboratory frame,

$$\begin{aligned} \mathbf{B}_0 &= \frac{1}{m} \left[q^0 \mathbf{B} - [\mathbf{q} \times \mathbf{E}] - \frac{\mathbf{q}(\mathbf{q} \cdot \mathbf{B})}{q^0 + m} \right] \\ &= \frac{1}{m} \left[q^0 \mathbf{H} - [\mathbf{q} \times \mathbf{D}] - \frac{\mathbf{q}(\mathbf{q} \cdot \mathbf{H})}{q^0 + m} + \frac{\mathbf{q}}{2\mu_0} \left(f^0 - \frac{(\mathbf{q} \cdot \mathbf{f})}{q^0 + m} \right) \right] - \frac{\mathbf{f}}{2\mu_0}. \end{aligned} \quad (5.11)$$

We see that the neutrino spin precesses around the direction \mathbf{B}_0 with the frequency $\omega = 2m|\mathbf{B}_0|/q^0$, the angle between \mathbf{B}_0 and the vector of spin being $\vartheta = \arccos(\boldsymbol{\zeta}_0 \cdot \boldsymbol{\zeta}_{\text{tp}})$. The spin vector direction corresponding to stationary states $\boldsymbol{\zeta}_{\text{tp}}$ is connected to the effective magnetic field as follows:

$$\boldsymbol{\zeta}_{\text{tp}} = \frac{\mathbf{B}_0}{|\mathbf{B}_0|}. \quad (5.12)$$

This fact explains in a simple way the stationarity of states with $S_0^\mu(q) = S_{\text{tp}}^\mu(q)$.

Let us introduce a flight length L and a spin oscillation length L_{osc} of the particle, remember that these are related by $\theta = \pi L/L_{\text{osc}}$ and that the scalar product $(Nx) = \tau$ may be interpreted as the proper time of a particle. The spin oscillation length is

$$L_{\text{osc}} = \frac{2\pi|\mathbf{q}|}{\sqrt{(fq)^2 - f^2m^2 - 4mH^{\mu\nu}f_\mu q_\nu + 4H^{\mu\alpha}H_{\alpha\nu}q_\mu q^\nu}}. \quad (5.13)$$

We can now write the probability for the neutrino which was arisen with polarization ζ_0 to change the polarization to $-\zeta_0$ after traveling some distance L . This is

$$W = \sin^2 \vartheta \sin^2(\pi L/L_{osc}). \quad (5.14)$$

It is clear that when $(\zeta_0 \cdot \zeta_{tp}) = 0$ or $\vartheta = \pi/2$, the probability can be equal to unity and the resonance takes place.

If the neutrino has the fixed helicity in the initial state

$$S_0^\mu(q) = \frac{1}{m} \{|\mathbf{q}|, q^0 \mathbf{q}/|\mathbf{q}|\}, \quad \zeta_0 = \frac{\mathbf{q}}{|\mathbf{q}|}, \quad (5.15)$$

then $\sin^2 \vartheta = 1 - (\mathbf{B}_0 \cdot \mathbf{q})^2/|\mathbf{B}_0|^2|\mathbf{q}|^2$ and formula (5.14) simplifies to the result widely discussed in the literature. This fact is not surprising. As was mentioned in Introduction, the Schrödinger-type equation with an effective Hamiltonian, which was used in papers [6] for obtaining this result, is merely the BMT equation in the spinor representation.

VI. DISCUSSION AND CONCLUSIONS

Let us compare our results with those of the standard quantum-mechanical approach to this problem. Since the neutrino behavior in dense matter under the influence of magnetic field, to the best of our knowledge, was not investigated before, we consider the neutrino propagation in a constant homogeneous magnetic field alone.

In the studies of the influence of a stationary pure magnetic field on the neutrino spin rotation in the pioneer paper [15], as well as in other papers, the stationary solutions $\Psi_{p\zeta}(x)$ first found in [42] were used as the wave functions of a particle. These solutions are the eigenfunctions of the canonical momentum operator p^μ and of the spin projection operator \mathfrak{S}_{tp} (see (4.12)). The description of the neutrino spin rotation there is based on solving the Cauchy problem where the initial condition is chosen in such a way that the mean value of neutrino helicity is equal to ± 1 . It was taken for granted that the solution of the Cauchy problem can be expressed as a linear combination of the above-mentioned wave functions:

$$\Psi(x) = \sum_{\zeta=\pm 1} c_\zeta(p) \Psi_{p\zeta}(x). \quad (6.1)$$

However, such an assumption is incorrect. The point is that, once in a pure state the mean value of some spin operator is equal to ± 1 , then this state is described by an eigenfunction of this operator. In the general case, the construction of the eigenfunction of the spin projection operator as a superposition of only positive-energy solutions of equation (1.3) is possible only when this spin projection operator commutes with the operator of the sign of the energy. The standard helicity operator $(\boldsymbol{\Sigma} \cdot \mathbf{p})/|\mathbf{p}|$ does not feature it.

The given phenomenon is a sort of the famous Klein paradox [43]. To avoid the indicated difficulties, in relativistic quantum mechanics only self-adjoint operators in the subspace of wave functions with a fixed energy sign can be treated as operators of observables. The choice of integrals of motion as operators of observables is the necessary condition to satisfy this requirement [24].

In the case considered the canonical momentum operator is an integral of motion. However, the conserved operator of the spin projection which should set initial conditions to the Cauchy problem is uniquely — up to the sign — determined by the form of the Dirac–Pauli equation. This operator is \mathfrak{S}_{tp} . Therefore, it is impossible to construct a wave function describing a neutrino with rotating spin in the form of an eigenfunction of the canonical momentum operator for its arbitrary eigenvalues. The solutions similar to (6.1) can exist when the special values of the canonical momentum are chosen. So, if a neutrino moves parallel or perpendicular to a constant homogeneous magnetic field, eigenfunctions of the helicity operator are the superpositions of positive-energy solutions alone [44].

To solve the problem we abandon the view that eigenvalues of the canonical momentum operator always impose a direction of the particle propagation. We found a self-adjoint operator \mathbf{p}^μ which can be interpreted as kinetic momentum operator of the particle and obtained the complete orthonormal system of the solutions of equation (1.3) with elements which are eigenfunctions of the given operator. On the base of this system we constructed solutions describing the neutrino with rotating spin. So our results enable one to treat a possible effect of the neutrino polarization change as a real precession of the particle spin.

Consequently, the problem of neutrino spin rotation in dense matter and in strong electromagnetic fields is solved in full agreement with the basic principles of quantum mechanics. Using the wave functions of orthonormal basis (3.24) or spin-coherent wave functions (3.21), it is possible to calculate probabilities of various processes with the neutrino in the framework of the Furry picture. When choosing one or another type of the basis, it is necessary to take into account that, due to the time-energy uncertainty, stationary states of the neutrino can be generated only when the linear size of the area occupied by the electromagnetic field and the matter is comparable in the order of magnitude with the formation length of the process — the spin oscillation length in our case.

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Appendix A

Let us find the equation for the description of the neutrino behavior in dense matter and in electromagnetic fields at low-energy limit. When weak interaction with background fermions is considered to be coherent, the behavior of mass states of any one-half spin lepton should be described by the Dirac-type equation,

$$(i\gamma^\mu \partial_\mu + \mathfrak{V}_{\text{em}} + \mathfrak{V}_{\text{matter}} - m)\Psi(x) = 0. \quad (\text{A1})$$

In this equation the term \mathfrak{V}_{em} describes interaction of the particle with the electromagnetic field and the term $\mathfrak{V}_{\text{matter}}$ is responsible for weak interaction with matter.

Following paper [45], the nature of the interaction terms is determined by the restrictions that the equation be Lorentz covariant and gauge invariant; that the terms are linear in the electromagnetic fields and integral characteristics of matter, i.e. currents and polarizations of background particles; that terms do not vanish in the limit of vanishing momentum of the particle; that the charge and current distribution associated with the particle be sufficiently localized that its interaction with slowly varying electromagnetic fields and characteristics of matter may be expressed in terms of the electromagnetic and matter potentials and arbitrary high derivatives of these potentials evaluated at the position of the particle. These assumptions lead to the term \mathfrak{V}_{em} in the form

$$\mathfrak{V}_{\text{em}} = - \sum_{n=0}^{\infty} \left[\varepsilon_n \gamma^\mu \square^n A_\mu + \frac{i}{2} \mu_n \sigma^{\mu\nu} \square^n F_{\mu\nu} + \mu'_n \gamma^5 (\gamma^\mu \square - \gamma^\nu \partial_\nu \partial^\mu) \square^n A_\mu + \frac{1}{2} \varepsilon'_n \gamma^5 \sigma^{\mu\nu} \square^n F_{\mu\nu} \right]. \quad (\text{A2})$$

Here A^μ is the potential of the external electromagnetic field, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor, $\square = \partial^\mu \partial_\mu$ is the d'Alembert operator. The constants $\mu_n, \mu'_n, \varepsilon_n, \varepsilon'_n$ characterize the interaction, ε_0 is charge of the particle, μ_0, ε'_0 are, respectively, anomalous magnetic and electric moments, and μ'_0 is an anapole moment. The expression for the $\mathfrak{V}_{\text{matter}}$ can be found if we replace A_μ in (A2) with a linear combination of the currents (1.1) and of the polarizations (1.2) of background fermions f with the proper choice of coupling constants.

We have the minimal nontrivial generalization of the Dirac–Pauli equation neglecting terms with derivatives higher than the second in (A2). In the expression for $\mathfrak{V}_{\text{matter}}$ we must hold only leading terms due to the proportionality of $\square A^\mu$ to the sum of charged particle currents. Further restrictions for (A1) depend on the sort of lepton and on the model of interaction. The neutrino is a neutral particle, thus $\varepsilon_0 = 0$. In the framework of the standard model where it is assumed that the theory is T invariant and the neutrino interacts with leptons and quarks through left currents, its anomalous electric moment goes to zero ($\varepsilon'_0 = 0$) and the term describing direct interaction of the neutrino with the currents contains multiplier $(1 + \gamma^5)$. As a result we come to equation (1.3) with an effective four-potential f^μ which is determined by (1.4) and (1.5).

Appendix B

For an arbitrary antisymmetric tensor $A^{\mu\nu}$, its dual tensor $*A^{\mu\nu} = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} A_{\rho\lambda}$, and for any four-vectors g^μ, h^μ , such as $(gh) \neq 0$ the following relation takes place [46]:

$$A^{\mu\nu}(gh) = -[g^\mu A^{\nu\rho} h_\rho - A^{\mu\rho} h_\rho g^\nu] + *[h^\mu *A^{\nu\rho} g_\rho - *A^{\mu\rho} g_\rho h^\nu]. \quad (\text{B1})$$

This leads to the formula

$$\begin{aligned} & g_\mu *A^\mu_\rho *A^\rho_\nu g^\nu h^2 + (g_\mu *A^\mu_\nu h^\nu)^2 - h_\mu A^\mu_\rho A^\rho_\nu h^\nu g^2 - (g_\mu A^\mu_\nu h^\nu)^2 = \\ & = g_\mu *A^\mu_\rho *A^\rho_\nu h^\nu(gh) - g_\mu A^\mu_\rho A^\rho_\nu h^\nu(gh) = 2(gh)^2 I_1. \end{aligned} \quad (\text{B2})$$

Here $I_1 = \frac{1}{4} A^{\mu\nu} A_{\mu\nu} = -\frac{1}{4} *A^{\mu\nu} *A_{\mu\nu}$ is the first invariant of the tensor $A^{\mu\nu}$.

Let $A^{\mu\nu} = {}^*F^{\mu\nu}$, $g^\mu = \varphi^\mu$, and $h^\mu = q^\mu$. Since ${}^*F^{\mu\nu}\varphi_\nu = -F^{\mu\nu}\varphi_\nu = 0$ (see (3.15), (3.16)), then from (B1) and (B2) we get

$$2H^{\mu\nu}(\varphi q) = H^{\mu\rho}q_\rho f^\nu - f^\mu H^{\nu\rho}q_\rho = m(\varphi^\mu f^\nu - f^\mu \varphi^\nu), \quad (\text{B3})$$

and

$$\begin{aligned} m^2((f\varphi)^2 - f^2\varphi^2) &= f^2\mathcal{N} + (f_\mu H^{\mu\nu}q_\nu)^2 = (fq)f_\mu H^{\mu\alpha}H_{\alpha\nu}q^\nu = 2(fq)^2\mu_0^2I_1, \\ \mathcal{N} - \tilde{\mathcal{N}} &= 2m^2\mu_0^2I_1. \end{aligned} \quad (\text{B4})$$

Let $A^{\mu\nu} = {}^*F^{\mu\nu}$, $g^\mu = \tilde{\Phi}_\mu$, and $h^\mu = \tilde{P}^\mu$. Since ${}^*F^{\mu\nu}\tilde{\Phi}_\nu = -F^{\mu\nu}\tilde{\Phi}_\nu = 0$, then from (B1), (B2) we get

$$2H^{\mu\nu}(\tilde{\Phi}\tilde{P}) = H^{\mu\rho}\tilde{P}_\rho f^\nu - f^\mu H^{\nu\rho}\tilde{P}_\rho = m(\tilde{\Phi}^\mu f^\nu - f^\mu \tilde{\Phi}^\nu), \quad (\text{B5})$$

and

$$m^2((f\tilde{\Phi})^2 - f^2\tilde{\Phi}^2) = f^2\tilde{P}_\mu H^{\mu\alpha}H_{\alpha\nu}\tilde{P}^\nu + (f_\mu H^{\mu\nu}\tilde{P}_\nu)^2 = (f\tilde{P})f_\mu H^{\mu\alpha}H_{\alpha\nu}\tilde{P}^\nu = 2(f\tilde{P})^2\mu_0^2I_1. \quad (\text{B6})$$

It is possible to establish by direct calculations using (B4) that

$$\frac{(\tilde{P}\tilde{\Phi})}{(q\varphi)} = \frac{(f\tilde{\Phi})}{(f\varphi)} = \left(1 + \zeta \frac{f_\mu H^{\mu\nu}q_\nu/2m - 2\mu_0^2I_1}{\sqrt{(\varphi q)^2 - m^2\varphi^2}}\right). \quad (\text{B7})$$

It follows from (B3) and (B5) that

$$\frac{\varphi^\mu}{(\varphi q)} = \frac{\tilde{\Phi}^\mu}{(\tilde{\Phi}\tilde{P})}, \quad (\text{B8})$$

and from (B4), (B6), and (B7) that

$$\begin{aligned} (\varphi q)^2 - m^2\varphi^2 &= \frac{(\varphi q)^2}{(\tilde{\Phi}\tilde{P})^2} [(\tilde{\Phi}\tilde{P})^2 - m^2\tilde{\Phi}^2], \\ \frac{(\varphi q)}{\sqrt{(\varphi q)^2 - m^2\varphi^2}} &= \Delta \frac{(\tilde{\Phi}\tilde{P})}{\sqrt{(\tilde{\Phi}\tilde{P})^2 - m^2\tilde{\Phi}^2}}. \end{aligned} \quad (\text{B9})$$

Appendix C

The Lorentz equation for the four-velocity u^μ and the BMT equation for the spin vector S^μ are

$$\begin{aligned} \dot{u}^\mu &= \frac{e}{m}F^{\mu\nu}u_\nu, \\ \dot{S}^\mu &= \frac{e}{m}F^{\mu\nu}S_\nu + 2\mu_0(g^{\mu\lambda} - u^\mu u^\lambda)F_{\lambda\nu}S^\nu. \end{aligned} \quad (\text{C1})$$

Since $u^2 = 1$ and $(Su) = 0$, we may rewrite these equations as [28]

$$\dot{u}^\mu = \Omega^{\mu\nu}u_\nu, \quad \dot{S}^\mu = \Omega^{\mu\nu}S_\nu, \quad (\text{C2})$$

where

$$\Omega^{\mu\nu} = \frac{e}{m}F^{\mu\nu} + 2\mu_0(g^{\mu\alpha} - u^\mu u^\alpha)F_{\alpha\beta}(g^{\beta\nu} - u^\beta u^\nu) \quad (\text{C3})$$

is an antisymmetric tensor.

From (C2) it is obvious that the evolution operators for the Lorentz and BMT equations are the same and fulfill the relation

$$\dot{R}^{\mu\nu}(\tau, \tau_0) = \Omega^\mu{}_\alpha R^{\alpha\nu}(\tau, \tau_0). \quad (\text{C4})$$

From (3.5) and (3.6) it follows that

$$j_V^\mu(\tau) = R^\mu{}_\nu(\tau, \tau_0) j_V^\nu(\tau_0) = \bar{\Psi}(\tau) \gamma^\mu \Psi(\tau) = \bar{\Psi}(\tau_0) U(\tau_0, \tau) \gamma^\mu U(\tau, \tau_0) \Psi(\tau_0), \quad (\text{C5})$$

and

$$j_A^\mu(\tau) = R^\mu{}_\nu(\tau, \tau_0) j_A^\nu(\tau_0) = \bar{\Psi}(\tau) \gamma^5 \gamma^\mu \Psi(\tau) = \bar{\Psi}(\tau_0) U(\tau_0, \tau) \gamma^5 \gamma^\mu U(\tau, \tau_0) \Psi(\tau_0). \quad (\text{C6})$$

Therefore, the equation for the evolution operator of the quasiclassical spin wave function takes the form

$$\dot{U}(\tau, \tau_0) = Z U(\tau, \tau_0), \quad (\text{C7})$$

where Z must obey the relations

$$[\gamma^\mu, Z] = \Omega^{\mu\nu} \gamma_\nu, \quad [\gamma^5 \gamma^\mu, Z] = \Omega^{\mu\nu} \gamma^5 \gamma_\nu. \quad (\text{C8})$$

It is obvious that

$$Z = \frac{1}{4} \Omega_{\mu\nu} \sigma^{\mu\nu}. \quad (\text{C9})$$

Using the relations

$$\frac{1}{2} F^{\mu\nu} \sigma_{\mu\nu} = i \gamma^5 {}^* F^{\mu\nu} u_\nu \gamma_\mu u^\alpha \gamma_\alpha + F^{\mu\nu} u_\nu \gamma_\mu u^\alpha \gamma_\alpha, \quad (\text{C10})$$

and

$$F^{\mu\nu} \sigma_{\mu\nu} = i \gamma^5 {}^* F^{\mu\nu} \sigma_{\mu\nu}, \quad (\text{C11})$$

we find

$$Z = i \gamma^5 \left(\frac{e}{4m} F^{\mu\nu} \gamma_\mu \gamma_\nu + \mu_0 F^{\mu\nu} u_\nu \gamma_\mu \gamma^\alpha u_\alpha \right). \quad (\text{C12})$$

We must take into account that the electric charge of neutrino e is equal to zero. Replacing $\mu_0 {}^* F^{\mu\nu}$ to $\mu_0 {}^* F^{\mu\nu} + (f^\mu u^\nu - u^\mu f^\nu)/2$ (see (3.3) and (3.4)), and introducing the notation $q^\mu = m u^\mu$ we obtain equation (3.7).

Appendix D

Let us prove the system (3.24) is orthonormal, i.e.

$$\mathbb{N}_+^2 = \int d\mathbf{x} \Psi_{q'\zeta'}^\dagger(x) \Psi_{q\zeta}(x) = (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') \delta_{\zeta\zeta'} \quad (\text{D1})$$

for the solutions with the same signs of energy and

$$\mathbb{N}_-^2 = \int d\mathbf{x} \Psi_{q'\zeta'}^\dagger(x) \Psi_{q\zeta}(x) = 0 \quad (\text{D2})$$

for the solutions with the different signs of energy. After integration we get

$$\mathbb{N}_+^2 = (2\pi)^3 \sqrt{J_{\zeta'}(\pm q')} \sqrt{J_\zeta(\pm q)} \left(\psi_{\zeta'}^\pm(q') \right)^\dagger \psi_\zeta^\pm(q) \delta^3(\mathbf{P}_{\zeta'}(\pm q') - \mathbf{P}_\zeta(\pm q)), \quad (\text{D3})$$

$$\mathbb{N}_-^2 = (2\pi)^3 \sqrt{J_{\zeta'}(\pm q')} \sqrt{J_\zeta(\mp q)} \left(\psi_{\zeta'}^\pm(q') \right)^\dagger \psi_\zeta^\mp(q) \delta^3(\mathbf{P}_{\zeta'}(\pm q') - \mathbf{P}_\zeta(\mp q)), \quad (\text{D4})$$

where the conventional eigenspinors $\psi_\zeta^\pm(q)$ obey the equations $(\gamma^\mu q_\mu \mp m) \psi_\zeta^\pm(q) = 0$; $J_\zeta(q)$ is the Jacobian for transition between the variables q^μ and $P_\zeta^\mu(q)$.

In the standard representation for the gamma matrices, we obtain explicitly [47]

$$\begin{aligned}\psi_{\zeta}^{+}(q) &= \frac{1}{\sqrt{2q^0(q^0+m)}} \begin{pmatrix} (q^0+m)\omega_{\zeta}^{+} \\ (\boldsymbol{\sigma} \cdot \mathbf{q})\omega_{\zeta}^{+} \end{pmatrix}, \\ \psi_{\zeta}^{-}(q) &= \frac{1}{\sqrt{2q^0(q^0+m)}} \begin{pmatrix} (\boldsymbol{\sigma} \cdot \mathbf{q})\omega_{\zeta}^{-} \\ (q^0+m)\omega_{\zeta}^{-} \end{pmatrix},\end{aligned}$$

where σ_i are the Pauli matrices and factor $1/\sqrt{2q^0(q^0+m)}$ has been included for normalization (3.11). These spinors satisfy the orthogonality relations

$$\left(\psi_{\zeta}^{\pm}(q)\right)^{\dagger} \gamma^0 \psi_{\zeta'}^{\mp}(q) = 0, \quad \left(\psi_{\zeta}^{\pm}(q)\right)^{\dagger} \psi_{\zeta'}^{\pm}(q) = \delta_{\zeta\zeta'}, \quad (\text{D5})$$

if ω_{ζ}^{\pm} are nonvanishing, but otherwise arbitrary two-component spinors which are chosen such that $(\omega_{\zeta'}^{\pm})^{\dagger} \omega_{\zeta}^{\pm} = \delta_{\zeta\zeta'}$. It is convenient to use the remaining uncertainty in ω_{ζ}^{\pm} to require $\psi_{\zeta}^{\pm}(q)$ to be eigenstates of the spin projection operator

$$-\gamma^5 \gamma_{\mu} S_{\text{tp}}^{\mu}(q) \psi_{\zeta}^{\pm}(q) = \zeta \psi_{\zeta}^{\pm}(q). \quad (\text{D6})$$

For this purpose ω_{ζ}^{\pm} should be eigenspinors of three-dimensional spin projection operator (see (5.4))

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\zeta}_{\text{tp}}) \omega_{\zeta}^{\pm} = \pm \zeta \omega_{\zeta}^{\pm}. \quad (\text{D7})$$

To calculate spinors ω_{ζ}^{\pm} we have an opportunity to choose a special reference frame. Let us take the reference frame where $f^{\mu} = \{f, 0, 0, 0\}$, $\mathbf{H} = (0, 0, H)$, and $\mathbf{E} = 0$. It is possible, if f^{μ} is timelike four-vector and the relation $F^{\mu\nu} f_{\nu} = 0$ is fulfilled.

In this reference frame the vector of the total polarization is

$$S_{\text{tp}}^{\mu}(q) = \frac{1}{mR_{\pm}} \left\{ q_{\perp} + q_{\parallel} \tilde{q}_{\pm}/q_{\perp}, q^0 \cos \phi, q^0 \sin \phi, q^0 \tilde{q}_{\pm}/q_{\perp} \right\}, \quad (\text{D8})$$

and

$$\boldsymbol{\zeta}_{\text{tp}} = \frac{1}{q_{\perp} R_{\pm}} \left\{ \mathbf{q} \left[1 \mp \frac{2\mu_0 H q_{\parallel}}{f(q^0+m)} \right] \pm 2\mu_0 \mathbf{H} \frac{q^0}{f} \right\}, \quad (\text{D9})$$

where

$$R_{\pm} = \text{sign}(f) \sqrt{(1 + 4\mu_0^2 H^2 / f^2) + (\tilde{q}_{\pm}/q_{\perp})^2}. \quad (\text{D10})$$

Here we use the notation

$$\sqrt{(q^1)^2 + (q^2)^2} = q_{\perp}, \quad q^1 = q_{\perp} \cos \phi, \quad q^2 = q_{\perp} \sin \phi, \quad q^3 = q_{\parallel}, \quad q_{\parallel} \pm 2\mu_0 m H / f = \tilde{q}_{\pm}. \quad (\text{D11})$$

The explicit form of spinors ω_{ζ}^{\pm} is

$$(q^0+m)\omega_{\zeta}^{\pm} = \frac{1}{T} \begin{pmatrix} e^{-i\phi/2} [(q^0+m) \mp 2\mu_0 H q_{\parallel}/f] \\ e^{i\phi/2} [(q^0+m)(\pm \zeta R_{\pm} - \tilde{q}_{\pm}/q_{\perp}) \mp 2\mu_0 H q_{\perp}/f] \end{pmatrix}, \quad (\text{D12})$$

$$(\boldsymbol{\sigma} \cdot \mathbf{q}) \omega_{\zeta}^{\pm} = \frac{1}{T} \begin{pmatrix} e^{-i\phi/2} [\pm \zeta R_{\pm} q_{\perp} \mp 2\mu_0 H q^0/f] \\ e^{i\phi/2} [q_{\perp} - q_{\parallel}(\pm \zeta R_{\pm} - \tilde{q}_{\pm}/q_{\perp})] \end{pmatrix},$$

where

$$T_{\pm} = \left\{ 2R_{\pm} \left[R_{\pm} \mp \zeta \left(\frac{\tilde{q}_{\pm}}{q_{\perp}} \pm \frac{2\mu_0 H q_{\perp}}{f(q^0+m)} \right) \right] \right\}^{1/2}. \quad (\text{D13})$$

In the selected reference frame the components of canonical momentum for positive-energy and for negative-energy solutions, respectively, are

$$\begin{aligned} P_\zeta^1(\pm q) &= \pm q_\perp \cos \phi \left(1 + \zeta \frac{f \mp 2\mu_0 H q_\parallel / m}{2q_\perp R_\pm} \right), \\ P_\zeta^2(\pm q) &= \pm q_\perp \sin \phi \left(1 + \zeta \frac{f \mp 2\mu_0 H q_\parallel / m}{2q_\perp R_\pm} \right), \\ P_\zeta^3(\pm q) &= \pm \tilde{q}_\pm \left(1 + \zeta \frac{f \mp 2\mu_0 H q_\parallel / m}{2q_\perp R_\pm} \right) - 2\mu_0 m H / f. \end{aligned} \quad (\text{D14})$$

Consider at first the $\zeta' = \zeta$ case. In this case on the right-hand side of equation (D3) we have

$$\delta^3(\mathbf{P}_\zeta(\pm q') - \mathbf{P}_\zeta(\pm q)) = |J_\zeta(\pm q)|^{-1} \delta^3(\mathbf{q}' - \mathbf{q}), \quad (\text{D15})$$

and as far as $q'^2 = q^2 = m^2$ we get $q'^\mu = q^\mu$. Therefore,

$$\mathbb{N}_+^2 = (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}'). \quad (\text{D16})$$

In equation (D4) the delta function provides the relation

$$\mathbf{q} + \mathbf{q}' = 0. \quad (\text{D17})$$

So we have

$$\mathbb{N}_-^2 = 0. \quad (\text{D18})$$

Consider next the $\zeta' \neq \zeta$ case. The delta function provides the following relations:

$$\tilde{q}'_\pm / q'_\perp = \tilde{q}_\pm / q_\perp, \quad \phi' = \phi, \quad (\text{D19})$$

on the right-hand side of equation (D3) and relations

$$\tilde{q}'_\pm / q'_\perp = -\tilde{q}_\mp / q_\perp, \quad \phi' = \phi + \pi, \quad (\text{D20})$$

on the right-hand side of equation (D4).

Using (D19) and (D20) and bearing in mind that $\zeta' = -\zeta$ we find

$$\left(\psi_{-\zeta}^\pm(q') \right)^\dagger \psi_\zeta^\pm(q) = 0, \quad \left(\psi_{-\zeta}^\pm(q') \right)^\dagger \psi_\zeta^\mp(q) = 0. \quad (\text{D21})$$

Thus we have proved relations (D1) and (D2), i.e. the orthogonality of system (3.24). It is easy to verify that for the cases where four-vector f^μ is spacelike or lightlike we can get the same result.

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$$Af(x) = \frac{1}{(2\pi)^n} \iint e^{i((x-y)\xi)} a(x, y, \xi) f(y) d^n y d^n \xi.$$

Pseudodifferential operators are extended to Fourier integral operators by

$$Ff(x) = \frac{1}{(2\pi)^n} \iint e^{i\phi(x, y, \xi)} a(x, y, \xi) f(y) d^n y d^n \xi.$$

In more detail see, for example, F. Trèves, *Introduction to Pseudodifferential and Fourier Integral Operators* (Plenum Press, New York, 1982).

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